

# N=2 supersymmetric particle near extreme Kerr throat

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## Abstract

We construct a new  $N = 2$  supersymmetric extension of a massive particle moving near the horizon of the extreme Kerr black hole. Our supercharges and Hamiltonian contain the proper number of fermions (two for each bosonic variables). The key ingredient of our construction is a proper choice of the bosonic variables which all have a clear geometric meaning.

# 1 Introduction

In the recent paper [1] the  $N = 2$  supersymmetric extension of a massive particle moving near the horizon of the extreme Kerr black hole has been constructed. The corner stone of this paper is the new solution of the vacuum Einstein equations found by Bardeen and Horowitz [2]. This solution describes the near horizon geometry of the extreme Kerr black hole. Very similarly to the extreme Reissner-Nordström black hole and its near horizon limit, the Bertotti-Robinson solution, which has an enhanced isometry group, the solution constructed in [2] also has an enhanced isometry group – the  $SO(1, 2) \times U(1)$  one. The appearance of the  $SO(1, 2)$  subgroup, which is just a conformal group in one dimension, is very important – one may conjecture that there is some dual conformal field theory description of this extreme Kerr black hole.

Motivated by the fact that the enhanced isometry group  $SO(1, 2) \times U(1)$  of the Bardeen and Horowitz solution coincides with the bosonic part of the  $N = 2$  superconformal group in one dimension, a  $N = 2$  supersymmetric extension of the massive particle moving in this background has been derived<sup>1</sup>. The explicit form of the supercharges and the corresponding Hamiltonian presented in [1] immediately raise the question – how can it be that the number of fermionic components is just two, while in the theory we have three bosonic fields? This fact is in contradiction with all known one-dimensional supersymmetric systems, in which the number of fermionic components is always larger than (or equal to) the number of bosonic ones. Nevertheless, the supercharges in [1] perfectly commute to span, together with the Hamiltonian and  $U(1)$  current, the  $N = 2$  superconformal group  $SU(1, 1|1)$ .

In the present paper we resolve this paradox and construct a new variant of the  $N = 2$  supersymmetric massive particle moving near the horizon of the extreme Kerr black hole. Roughly speaking, the situation looks as follows. Firstly, for any one-dimensional conformal invariant system there is a preferable coordinate system in which the Hamiltonian reads [5]:

$$H = \frac{P_R^2}{2} + \frac{J}{R^2}, \quad (1.1)$$

where  $R$  is a “radial” variable<sup>2</sup>,  $P_R$  is the corresponding momentum, while  $J$  describes the “angular” part of the system. The form of the Hamiltonian in (1.1) is the visualization of the commonly known fact that any system with the Hamiltonian  $J$  can be made conformal invariant by introducing a proper coupling with the dilaton. Clearly, the “angular” and “radial” parts in (1.1) are completely independent

$$\{P_R, J\} = \{R, J\} = 0. \quad (1.2)$$

Secondly, when we turn to derive a supersymmetric extension of the system in (1.1) there are two possibilities. The simplest one is to treat the “angular” part of the system  $J$  as a constant which does not participate in the supersymmetrization. With such approach any inner structure of the “angular” part is completely irrelevant – the final supercharges and Hamiltonian will have the same form depending on  $J$  as on a coupling constant. Just this variant of supersymmetrization has been constructed in [1]. It is worth to note that the resulting system is nothing but just a supersymmetric conformal mechanics. Moreover, one may increase the number of supersymmetries to four, or even to an arbitrary even number  $N = 2k$  [6, 7]. The corresponding superconformal group will be the  $SU(1, 1|k)$  one. One should stress that, despite the fact that such an extended supersymmetric system has a  $SO(1, 2) \times U(1) \times SU(k)$  bosonic subgroup, there is no contradiction with the isometry group  $SO(1, 2) \times U(1)$  of the background metric we started with: the additional group  $SU(k)$  is realized on the fermionic components only. It is just invisible while we are dealing with the bosonic sector only. Clearly, it is not possible to establish any link between supersymmetry charges and Killing spinors within the proposed supersymmetrization scheme, because the “angular” sector left untouched by supersymmetry.

Another variant of the supersymmetrization of the system with the Hamiltonian (1.1) includes the supersymmetrization of the “radial” as well as “angular” sectors of the model. In this case the restrictions imposed by the existence of the supercharges are stronger, and the maximal supersymmetry which can be achieved is indeed the  $N = 2$  one. The number of physical fermions in the resulting system will be six - two fermionic components per each bosonic ones, as it should be in the  $N = 2$  supersymmetric one-dimensional model. The construction of such a supersymmetric system is the subject of the present paper.

<sup>1</sup>A particle moving in Bertotti-Robinson space-time admits  $N = 4$  superconformal symmetry [3, 4].

<sup>2</sup>It is just equal to  $R = e^{\frac{u}{2}}$ , where  $u$  is a standard dilaton field

The paper is organized as follows. In section 2 we review the results obtained in [2, 1]. In addition we also present four additional multi-parameters solution of the vacuum Einstein equations which are very similar to those obtained in [2]. In section 3 we introduce new variables, in order to bring the bosonic Hamiltonian to the form (1.1). The supercharges and the Hamiltonian are derived in section 4. We conclude with a short discussion.

## 2 Preliminaries

The metric we are interesting in, has the form [2]<sup>3</sup>

$$ds^2 = f_1 \left[ -r^2 d\tau^2 + \frac{1}{r^2} dr^2 + d\theta^2 \right] + f_2 [d\phi + r d\tau]^2, \quad (2.1)$$

where

$$f_1 = \frac{1 + \cos^2 \theta}{2}, \quad f_2 = \frac{2 \sin^2 \theta}{1 + \cos^2 \theta}. \quad (2.2)$$

The metric (2.1) is the solution of the vacuum Einstein equations [2]. In principle, forgetting about the former source of the metric (2.1), one may find some additional metrics still obeying the vacuum Einstein equations. Indeed, if we let the functions  $f_1, f_2$  in (2.2) be arbitrary, depending on the coordinate  $\theta$  only, then, the condition that the metric (2.1) obeys the vacuum Einstein equations results in the following restrictions on these functions:

$$4f_1^2 + 3(f_1')^2 + 2f_1 f_1'' - 2(f_1'')^2 + 3f_1' f_1^{(3)} = 0, \quad (2.3)$$

$$f_2 = \frac{1}{3f_1} (4f_1^2 - 3(f_1')^2 + 4f_1 f_1''). \quad (2.4)$$

Among the solutions of the equation (2.3) there is subset which is similar to (2.2). This subset of solutions can be selected by the following Ansatz

$$f_1 = a_1 + a_2 \sin(\theta) + a_3 \cos(\theta) + a_4 \sin(2\theta) + a_5 \cos(2\theta) \quad (2.5)$$

which yields five solutions<sup>4</sup>

<i>sol</i>	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$
<i>I</i>	$-3\sqrt{a_4^2 + a_5^2}$	$-\frac{a_3 a_5}{a_4} - \frac{a_3 \sqrt{a_4^2 + a_5^2}}{a_4}$	$a_3$	$a_4$	$a_5$
<i>II</i>	$3\sqrt{a_4^2 + a_5^2}$	$-\frac{a_3 a_5}{a_4} + \frac{a_3 \sqrt{a_4^2 + a_5^2}}{a_4}$	$a_3$	$a_4$	$a_5$
<i>III</i>	0	$a_2$	$a_3$	0	0
<i>IV</i>	$3a_5$	0	$a_3$	0	$a_5$
<i>V</i>	$-3a_5$	$a_2$	0	0	$a_5$

(2.6)

The metric in (2.1) corresponds to solution IV with  $a_5 = \frac{1}{4}$  and  $a_3 = 0$ .

All these metrics share the same property – they are invariant with respect to  $SO(1, 2)$  transformations realized in an unusual way as [2]

$$\delta\tau = a + b\tau + c \left( \tau^2 + \frac{1}{r^2} \right), \quad \delta r = -b r - 2c \tau r, \quad \delta\phi = -\frac{2c}{r}, \quad \delta\theta = 0, \quad (2.7)$$

where  $a, b, c$  are the parameters of  $SO(1, 2)$ .

<sup>3</sup>We fixed one inessential parameter  $r_0$  originally present in [2] to be  $r_0^2 = 2M^2 = 1$ .

<sup>4</sup>For each solution with a given set of parameters one has check that  $f_2 \neq 0$ . Otherwise, the metric (2.1) will be degenerate.

Just the presence of this symmetry was the main motivation in the paper [1] where the static gauge action of a relativistic particle moving in the background with the metric (2.1) has been written as

$$S = -m \int \sqrt{-ds^2} = -m \int d\tau \sqrt{f_1 \left( r^2 - \frac{\dot{r}^2}{r^2} - \dot{\theta}^2 \right) - f_2 \left( \dot{\phi} + r \right)^2}. \quad (2.8)$$

With this action, the Hamiltonian, together with the first integrals generating the conformal transformations (2.7), were constructed as

$$\begin{aligned} H &= r \left( \sqrt{m^2 f_1 + (rp_r)^2 + p_\theta^2 + \frac{f_1}{f_2} p_\phi^2} - p_\phi \right), \\ K &= \frac{1}{r} \left( \sqrt{m^2 f_1 + (rp_r)^2 + p_\theta^2 + \frac{f_1}{f_2} p_\phi^2} + p_\phi \right) + \tau^2 H + 2\tau r p_r, \\ D &= r p_r + \tau H. \end{aligned} \quad (2.9)$$

Under the Poisson brackets

$$\{r, p_r\} = 1, \quad \{\phi, p_\phi\} = 1, \quad \{\theta, p_\theta\} = 1, \quad (2.10)$$

they form  $so(1, 2)$  algebra

$$\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K. \quad (2.11)$$

One additional integral of motion fully commuting with  $(H, D, K)$ , is  $p_\phi$ .

In the same paper [1] the  $N = 2$  supersymmetric extension of the action (2.8) has been constructed. In the subsequent sections we will construct the extended version of the supersymmetric mechanics of [1]. However, before passing to the supersymmetrization procedure it makes sense to change our coordinates (fields) to bring the  $SO(1, 2)$  transformations to a more conventional form.

## 3 New variables

### 3.1 Standard realization of conformal symmetry

Let us start with the  $so(1, 2)$  algebra

$$i[H, D] = -H, \quad i[K, D] = K, \quad i[H, K] = -2D \quad (3.1)$$

The conformal symmetry can be realized on the  $SO(1, 2)$  group element  $g$

$$g = e^{itH} e^{izK} e^{iuD} \quad (3.2)$$

by the left shifts

$$g_0 \cdot g = g', \quad g_0 = e^{iaH} e^{icK} e^{ibD} \quad (3.3)$$

as

$$\delta t = a + bt + ct^2, \quad \delta u = b + 2ct, \quad \delta z = c - (b + 2ct)z. \quad (3.4)$$

One should stress that the transformation laws of  $\dot{u}$

$$\delta \dot{u} = 2c - (b + 2ct) \dot{u} \quad (3.5)$$

coincide with those for  $2z$ . These are the conventional transformation properties of the fields under the one-dimensional conformal group [8].

One may extend the  $so(1, 2)$  algebra (3.1) by the additional generator  $U$ , which commutes with the set  $(H, D, K)$ . The extended group element  $\tilde{g}$  will contain the additional factor

$$\tilde{g} = e^{itH} e^{izK} e^{iuD} e^{i\theta U}, \quad (3.6)$$

and the new bosonic field  $\theta(t)$  will have trivial transformation properties under the  $SO(1, 2) \times U(1)$  group

$$\delta \theta = \gamma, \quad (3.7)$$

where  $\gamma$  is the parameter of a  $U(1)$  transformation.

All  $SO(1, 2)$  invariant objects can be constructed with the help of Cartan forms

$$\tilde{g}^{-1}d\tilde{g} = i\omega_H H + i\omega_K K + i\omega_D D + i\omega_U U, \quad (3.8)$$

where for our parametrization of  $\tilde{g}$  (3.6) the forms read

$$\omega_H = e^{-u} dt, \quad \omega_K = e^u (dz + z^2 dt), \quad \omega_D = du - 2z dt, \quad \omega_U = d\theta. \quad (3.9)$$

It is a matter of calculation to relate  $(t, u, z)$  with the coordinates  $(\tau, r, \phi)$

$$t = \tau + \frac{1}{r}, \quad u = -\log r - \phi, \quad z = \frac{r}{2}. \quad (3.10)$$

In these variables the metric (2.1) acquires the form

$$\begin{aligned} ds^2 &= f_1 [-4dt (dz + z^2 dt) + d\theta^2] + f_2 [du - 2z dt]^2 \\ &= f_1 [-4\omega_P \omega_K + \omega_U^2] + f_2 \omega_D^2. \end{aligned} \quad (3.11)$$

The full symmetry group leaving the metric (3.11) invariant, includes also one additional isometry

$$\delta u = a_0, \quad (3.12)$$

which commutes with  $SO(1, 2)$  transformations (2.7). It is worth to note that just this additional isometry almost completely fixed (up to the functions  $f_1, f_2$ ) the metric (3.11).

Let us remind that the standard conformal mechanics is defined by the action [8, 9]

$$S_{stand} = \int dt [\alpha \omega_H + \beta \omega_K + \gamma \omega_D], \quad (3.13)$$

where  $(\alpha, \beta, \gamma)$  are arbitrary constant parameters. In addition, the standard description is supplied by the Inverse Higgs condition [10]

$$\omega_D = 0 \quad \Rightarrow \quad z = \frac{1}{2} \dot{u}, \quad (3.14)$$

which expressed the additional Goldstone boson  $z(t)$  in terms of the dilaton  $u(t)$ . This is achieved without any breaking of conformal symmetry due to the transformation properties (3.5). In contrast, in the action (3.11) the boson  $z(t)$  itself is present. Of course, one may additionally impose the condition (3.14); however, after this we will end up with three two-dimensional systems. Thus, in what follows, we will avoid imposing any additional constraints. Note, that even in this case one may introduce, instead of the field  $z(t)$ , another one defined as

$$\tilde{z} = e^u \left( z - \frac{1}{2} \dot{u} \right) \quad \Rightarrow \quad \delta \tilde{z} = 0. \quad (3.15)$$

Clearly,  $\tilde{z}$  transforms as an ordinary scalar field, like  $\theta$ , under the conformal  $SO(1, 2)$  group.

Another way to construct the conformal mechanics is to use the following action

$$S = -\gamma \int \sqrt{\omega_H \omega_K} = -\gamma \int dt \sqrt{\dot{z} + z^2}. \quad (3.16)$$

The equation of motion for a single variable  $z$  reads

$$\ddot{z} + 6z\dot{z} + 4z^3 = 0. \quad (3.17)$$

The variable  $z(t)$  is not very convenient: it has the unusual dimension ( $cm^{-1}$ ) and is shifted by a constant under the conformal boost (3.4). This is the property of the Goldstone field for the conformal boost. Moreover, in higher dimensions this variable is a vector with respect to the Lorentz group. All this yields the motivation for introducing a new variable  $x(t)$  as

$$z(t) = \frac{d}{dt} \log x. \quad (3.18)$$

For this variable the equation of motion (3.17) will be of the third order in time derivatives

$$x\ddot{x} + 3\dot{x}\ddot{x} = 0. \quad (3.19)$$

Nevertheless, one may rewrite the equation (3.19) as

$$\frac{d}{dt}(x^3\ddot{x}) = 0 \quad \Rightarrow \quad x^3\ddot{x} = g^2 = \text{const.} \quad (3.20)$$

Keeping in mind that under the conformal group  $x$  transforms as

$$\delta x = \frac{1}{2}(b + 2ct)x, \quad (3.21)$$

one may check that the combination  $x^3\ddot{x}$  is a scalar under the conformal group and, therefore, the presence of the constant  $g^2$  in (3.20) preserves the conformal symmetry. Clearly, the equation (3.20) is just an equation of motion of the standard conformal mechanics [8].

Another way to get the same result is to impose the Inverse Higgs phenomenon condition (3.14) and then to represent the equation (3.17) as

$$\frac{d}{dt} \left[ e^{\frac{3}{2}u} \frac{d^2}{dt^2} \left( e^{\frac{1}{2}u} \right) \right] = 0 \quad \Rightarrow \quad \frac{d^2}{dt^2} \left( e^{\frac{1}{2}u} \right) = g^2 e^{-\frac{3}{2}u}. \quad (3.22)$$

Clearly, this is the same equation as (3.20) one, upon the identification of  $x$  with  $e^{\frac{1}{2}u}$ .

Thus, the action (3.16) provides an alternative description of conformal mechanics, when the coupling constant  $g$  appears as an integration constant.

### 3.2 Particle action in new coordinates

In newly defined coordinates (3.10) the action (2.8) acquires the form<sup>5</sup>

$$S = -m \int dt \sqrt{f_1 \left[ 4(\dot{z} + z^2) - \dot{\theta}^2 \right] - f_2 (\dot{u} - 2z)^2}. \quad (3.23)$$

The generators of conformal transformations read

$$\begin{aligned} H &= - \frac{f_1 (p_u^2 + m^2 f_2) + f_2 (p_\theta^2 - 2z p_u p_z + z^2 p_z^2)}{f_2 p_z}, \\ D &= -p_u + z p_z + t H, \quad K = -p_z + 2t D - t^2 H. \end{aligned} \quad (3.24)$$

They form the same  $so(1, 2)$  algebra (2.11) with respect to standard Poisson brackets

$$\{u, p_u\} = 1, \quad \{z, p_z\} = 1, \quad \{\theta, p_\theta\} = 1. \quad (3.25)$$

The Casimir element of the  $so(1, 2)$  algebra is simplified to be

$$J = 2 (H K - D^2) = 2 \left[ \left( \frac{f_1}{f_2} - 1 \right) p_u^2 + p_\theta^2 + m^2 f_1 \right]. \quad (3.26)$$

Now, following [5], we introduce the new canonical variable  $R(t)$  and its momentum  $P_R$  as

$$R = \sqrt{2K}, \quad P_R = -\frac{\sqrt{2}D}{\sqrt{K}} \quad \rightarrow \quad \{R, P_R\} = 1. \quad (3.27)$$

With the help of these variables one may rewrite our Hamiltonian (3.24) as

$$H = \frac{P_R^2}{2} + \frac{J}{R^2}. \quad (3.28)$$

This is the standard form of any conformal invariant mechanics [5]. Thus, the task of constructing a  $N = 2$  supersymmetric extension of our model is reduced to a more simple problem - the construction of the supersymmetric extension of the system with the Hamiltonian (3.28).

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<sup>5</sup>The action, similar to (3.23), has been recently considered in [11].

## 4 Supersymmetric extensions

### 4.1 Supersymmetrization of the “radial” variable

If we will treat our Casimir operator  $J$  (3.26) as a coupling constant, which does not participate in the supersymmetrization procedure, then the Hamiltonian  $H$  (3.28) is just the Hamiltonian of standard conformal mechanics [8]. In this case  $N = 2$  superconformal extension of our system has the same form as  $N = 2$  superconformal mechanics [12]

$$\begin{aligned} \mathcal{Q} &= \left( P_R + \frac{i\sqrt{2J}}{R} \right) \psi, \quad \bar{\mathcal{Q}} = \left( P_R - \frac{i\sqrt{2J}}{R} \right) \bar{\psi}, \quad \mathcal{S} = -R\psi, \quad \bar{\mathcal{S}} = -R\bar{\psi}, \\ \mathcal{H} &= \frac{P_R^2}{2} + \frac{J}{R^2} - \frac{\sqrt{2J}}{R^2} \psi \bar{\psi}, \quad \mathcal{D} = -\frac{1}{2} R P_R, \quad \mathcal{K} = \frac{1}{2} R^2, \quad \mathcal{U} = \sqrt{2J} + \psi \bar{\psi}. \end{aligned} \quad (4.1)$$

Here, we introduced the fermionic variables  $(\psi, \bar{\psi})$  obeying the following brackets

$$\{\psi, \bar{\psi}\} = i. \quad (4.2)$$

One may easily check that all commutators of  $N = 2$  superconformal algebra are satisfied<sup>6</sup>:

$$\begin{aligned} \{\mathcal{H}, \mathcal{D}\} &= \mathcal{H}, \quad \{\mathcal{K}, \mathcal{D}\} = -\mathcal{K}, \quad \{\mathcal{H}, \mathcal{K}\} = 2\mathcal{D}, \\ \left\{ \left( \frac{\mathcal{Q}}{\mathcal{Q}} \right), \mathcal{D} \right\} &= \frac{1}{2} \left( \frac{\mathcal{Q}}{\mathcal{Q}} \right), \quad \left\{ \left( \frac{\mathcal{S}}{\mathcal{S}} \right), \mathcal{D} \right\} = -\frac{1}{2} \left( \frac{\mathcal{S}}{\mathcal{S}} \right), \\ \left\{ \left( \frac{\mathcal{Q}}{\mathcal{Q}} \right), \mathcal{K} \right\} &= \left( \frac{\mathcal{S}}{\mathcal{S}} \right), \quad \left\{ \left( \frac{\mathcal{S}}{\mathcal{S}} \right), \mathcal{H} \right\} = -\left( \frac{\mathcal{Q}}{\mathcal{Q}} \right), \\ \left\{ \left( \frac{\mathcal{Q}}{\mathcal{Q}} \right), \mathcal{U} \right\} &= i \left( \frac{-\mathcal{Q}}{\mathcal{Q}} \right), \quad \left\{ \left( \frac{\mathcal{S}}{\mathcal{S}} \right), \mathcal{U} \right\} = i \left( \frac{-\mathcal{S}}{\mathcal{S}} \right), \\ \{\mathcal{Q}, \bar{\mathcal{Q}}\} &= 2i\mathcal{H}, \quad \{\mathcal{S}, \bar{\mathcal{S}}\} = 2i\mathcal{K}, \quad \{\mathcal{Q}, \bar{\mathcal{S}}\} = 2i\mathcal{D} + \mathcal{U}, \quad \{\bar{\mathcal{Q}}, \mathcal{S}\} = 2i\mathcal{D} - \mathcal{U}. \end{aligned} \quad (4.3)$$

This is the  $N = 2$  supersymmetrization constructed in [1].

Clearly, this variant of supersymmetrization does not feel any structure of the theory hidden in the Casimir  $J$ . This is just the supersymmetrization of the “radial” variable  $R$ . It is going to go in the same way for any  $J$ . The “angular” sector, defined by the explicit structure of  $J$ , can contain as many variables as we wish – nothing will change. Moreover, the number of supersymmetries is not restricted. Indeed, following [6, 7] one may easily construct the  $N = 2k$  superconformal invariant extension of this system for arbitrary  $k$  in a similar way. The corresponding superconformal group will be the  $SU(1, 1|k)$  one.

### 4.2 New variant of the supersymmetrization

Another variant of the supersymmetrization we are going to construct here, will include an additional  $N = 2$  supersymmetric extension of the Casimir  $J$ . The present variant crucially depends on the structure of the “angular” sector of the theory, defined by  $J$  (3.26). It is based on the same ideas as in [5]. Generally, the construction goes as follows.

First of all, let us rewrite our bosonic Hamiltonian (3.28) as

$$H = \frac{P_R^2}{2} + \frac{m^2}{R^2} + \frac{\tilde{J}}{R^2}, \quad \tilde{J} = J - m^2 \quad (4.4)$$

Let us further suppose that we were able to find a proper  $N = 2$  supersymmetric extension of the “angular Hamiltonian”  $J_{SUSY} = \tilde{J} + \text{fermions}$ . This means that there are two supercharges  $q, \bar{q}$  which anti-commute to produce  $J_{SUSY}$  (These supercharges will be constructed in the next Section)

$$\{q, \bar{q}\} = 2iJ_{SUSY} = 2i(\tilde{J} + \text{fermions}), \quad \{q, q\} = \{\bar{q}, \bar{q}\} = 0. \quad (4.5)$$

We will also need the  $U(1)$  generator  $U_A$  for the supercharges  $q, \bar{q}$  of the “angular” sector

$$\{U_A, q\} = iq, \quad \{U_A, \bar{q}\} = -i\bar{q}. \quad (4.6)$$

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<sup>6</sup>We write down only non-vanishing brackets

Clearly,  $U_A$  is precisely the  $R$ -symmetry generator of  $N = 2$  supersymmetry. For the concrete system it has a standard realization through the bilinear combinations of the fermions of the “angular” sector (see the next Section). In the general case, without going into the details of the particular model, one may build this generator in terms of the supercharges  $q, \bar{q}$  and the Hamiltonian  $J_{SUSY}$  as

$$U_A = \frac{q \bar{q}}{2 J_{SUSY}}. \quad (4.7)$$

Now, it is a matter of calculation to check that the following generators form the same  $N = 2$  superconformal algebra (4.3)

$$\begin{aligned} \tilde{Q} &= Q_0 + \frac{q}{R} + i \frac{U_A \psi}{R}, & \tilde{\bar{Q}} &= \bar{Q}_0 + \frac{\bar{q}}{R} - i \frac{U_A \bar{\psi}}{R}, & \tilde{S} &= -R\psi, & \tilde{\bar{S}} &= -R\bar{\psi} \\ \tilde{H} &= H_0 + \frac{\tilde{J}}{R^2} + \frac{1}{R^2} \left( \sqrt{2} m U_A - i \psi \bar{q} - i \bar{\psi} q - U_A \psi \bar{\psi} \right), \\ \tilde{U} &= \sqrt{2} m + \psi \bar{\psi} + U_A, & \tilde{K} &= \frac{1}{2} R^2, & \tilde{D} &= -\frac{1}{2} R P_R, \end{aligned} \quad (4.8)$$

where

$$Q_0 = \left( P_R + i \frac{\sqrt{2} m}{R} \right) \psi, \quad \bar{Q}_0 = \left( P_R - i \frac{\sqrt{2} m}{R} \right) \bar{\psi}, \quad H_0 = \frac{1}{2} P_R^2 + \frac{m^2}{R^2} - \frac{\sqrt{2} m \psi \bar{\psi}}{R^2}, \quad (4.9)$$

and, therefore,

$$\{Q_0, \bar{Q}_0\} = 2i H_0. \quad (4.10)$$

Thus, the present construction extends the arbitrary  $N = 2$  supersymmetric system with the Hamiltonian  $J_{SUSY}$  to the superconformal mechanics.

### 4.3 Supersymmetric extension of the “angular” sector

As it follows from the previous subsection, the  $N = 2$  superconformal extension of our system is reduced now to a much simpler task – the derivation of a  $N = 2$  supersymmetric extension of the “angular” sector with the bosonic Hamiltonian

$$\tilde{J} = 2 \left( \frac{f_1}{f_2} - 1 \right) p_u^2 + 2 p_\theta^2 + m^2 (2 f_1 - 1). \quad (4.11)$$

In order to find a proper supersymmetrization of the system with this bosonic Hamiltonian, let us firstly introduce a pair of complex fermionic variables  $\rho, \xi$  obeying the brackets

$$\{\rho, \bar{\rho}\} = i, \quad \{\xi, \bar{\xi}\} = i. \quad (4.12)$$

Now, it is rather easy to check that the supercharges  $q, \bar{q}$

$$q = 2 p_\theta \rho + 2 h_1 p_u \xi + i \sqrt{2} m h_2 \rho + i h_3 \xi \bar{\xi} \rho, \quad \bar{q} = 2 p_\theta \bar{\rho} + 2 h_1 p_u \bar{\xi} - i \sqrt{2} m h_2 \bar{\rho} - i h_3 \xi \bar{\xi} \bar{\rho}, \quad (4.13)$$

where the arbitrary, for the time being, functions  $h_1, h_2$  and  $h_3$  depend on  $\theta$  only, form the superalgebra (4.5) with the Hamiltonian

$$J_{SUSY} = 2 p_\theta^2 + 2 h_1^2 p_u^2 + m^2 h_2^2 + 4 i h_1' p_u (\rho \bar{\xi} + \bar{\rho} \xi) - 2 \sqrt{2} m \frac{h_1' h_2}{h_1} \xi \bar{\xi} + 2 \sqrt{2} m h_2' \rho \bar{\rho} - 4 \left( \frac{h_1'}{h_1} \right)' \rho \bar{\rho} \xi \bar{\xi}, \quad (4.14)$$

if the function  $h_3$  is defined as

$$h_3 = -2 \frac{h_1'}{h_1}. \quad (4.15)$$

Finally, identifying the bosonic part of  $J_{SUSY}$  (4.14) with  $\tilde{J}$  (4.11) we will fix all our functions to be

$$h_1 = \sqrt{\frac{f_1}{f_2} - 1}, \quad h_2 = \sqrt{2 f_1 - 1}. \quad (4.16)$$



It is worth to note that with the newly introduced fermions  $\rho, \xi$  the  $U_A$  current (4.6) can be easily constructed as

$$U_A = \rho \bar{\rho} + \xi \bar{\xi}. \quad (4.17)$$

Substituting the supercharges  $q, \bar{q}$  (4.13) and the  $U_A$  current (4.17) into the supercharges and the Hamiltonian of the full system (4.8), we will get the desired  $N = 2$  supersymmetric extension of our system. It has a proper number of fermions - two for each bosonic variables, as it should be in  $N = 2$  supersymmetric mechanics. One should stress that, with  $m = 0$ , the bosonic Hamiltonian  $\tilde{J}$  (4.11) can be supersymmetrized, with the help of only two fermions, within chiral superfields. However, for a generic value of  $m$ , this is impossible and we have to invoke four fermions - i.e. one complex fermions for each boson. The peculiarity of the given system is encoded in the explicit expression for the functions  $h_1, h_2, h_3$  entering the supercharges (4.13) and the Hamiltonian (4.14). In contrast, the supersymmetrization of only the “radial” variable does not feel the fine structure of the “angular” sector and it has the same form (4.1) for any  $\tilde{J}$ . That is the reason why the number of supersymmetries can be chosen arbitrarily for the variant of the supersymmetrization considered in [1].

## 5 Conclusion and Discussion

In the present paper, following [1], we constructed a new  $N = 2$  supersymmetric extension of a massive particle moving near the horizon of the extreme Kerr black hole. The main motivation for our paper was the unusual variant of supersymmetrization proposed in [1]. Indeed, the supercharges and the Hamiltonian constructed in [1] contain only two fermions, while the number of physical bosons is three, in contrast with any customary countings in one dimensional supersymmetric systems, which all claim that the number of physical fermions cannot be smaller than the number of physical bosons. We analyzed in details this situation and explicitly demonstrated that the supersymmetrization procedure used in [1] deals only with the “radial” sector of the system. Such a supersymmetrization leaves aside all peculiarities of the system encoded in its “angular” sector. We also chose a proper set of bosonic variables with clear geometric meaning, and then we constructed new  $N = 2$  supercharges and Hamiltonian which solved our task.

Clearly, the proposed approach can be applied to any system possessing  $SO(1, 2)$  symmetry, including four additional multi-parameters solutions of the vacuum Einstein equation we found in this paper. Clearly, the system recently presented in [13], which suffers from the same problems as that in [1], can be easily re-formulated within our approach.

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## References

- [1] A. Galajinsky,  
“Particle dynamics near extreme Kerr throat and supersymmetry”,  
JHEP 1011 (2010) 126, [arXiv:1009.2341\[hep-th\]](#).
- [2] J. Bardeen, G.T. Horowitz,  
“The Extreme Kerr throat geometry: A Vacuum analog of  $AdS(2) \times S^2$ ”,  
Phys.Rev. **D60** (1999) 104030; [arXiv:hep-th/9905099](#).
- [3] S. Bellucci, A. Galajinsky, E. Ivanov, S. Krivonos,  
“AdS(2)/CFT(1), canonical transformations and superconformal mechanics”,  
Phys.Lett. **B555** (2003) 99; [arXiv:hep-th/0212204](#).

- [4] A. Galajinsky,  
“Particle dynamics on  $AdS_2 \times S^2$  background with two-form flux”,  
Phys.Rev. **D78** (2008) 044014, [arXiv:0806.1629\[hep-th\]](#).
- [5] T. Hakobyan, S. Krivonos, O. Lechtenfeld, A. Nersessian,  
“Hidden symmetries of integrable conformal mechanical systems”,  
Phys.Lett. **A374** (2010) 801, [arXiv:0908.3290\[hep-th\]](#).
- [6] E.A. Ivanov, S.O. Krivonos, V.M. Leviant,  
“Geometric Superfield Approach To Superconformal Mechanics”,  
J.Phys. **A22** (1989) 4201.
- [7] T. Hakobyan, A. Nersessian,  
“Lobachevsky geometry of (super)conformal mechanics”,  
Phys.Lett. **A373** (2009) 1001, [arXiv:0803.1293\[hep-th\]](#).
- [8] V. de Alfaro, S. Fubini, G. Furlan,  
“Conformal Invariance in Quantum Mechanics”,  
Nuovo Cim. **A34** (1976) 569.
- [9] E.A. Ivanov, S.O. Krivonos, V.M. Leviant,  
“Geometry Of Conformal Mechanics”,  
J.Phys. **A22** (1989) 345.
- [10] E. Ivanov, V. Ogievetsky,  
“The Inverse Higgs Phenomenon in Nonlinear Realizations”,  
Teor.Mat.Fiz. **25** (1975) 164.
- [11] S. Fedoruk, E. Ivanov, J. Lukierski,  
“Galilean Conformal Mechanics from Nonlinear Realizations”,  
[arXiv:1101.1658\[hep-th\]](#).
- [12] V. Berezhovoj, A. Pashnev,  
“On the Structure of the N=4 Supersymmetric Quantum Mechanics in D=2 and D=3”,  
Class. Quant. Grav. **13** (1996) 1699, [arXiv:hep-th/9506094](#).
- [13] A. Galajinsky, K. Orekhov,  
“N=2 superparticle near horizon of extreme Kerr-Newman-AdS-dS black hole”,  
Nucl.Phys. **B850** (2011) 339, [arXiv:1103.1047\[hep-th\]](#).